Computing dynamical systems

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Theorem. When the program is started at 2, the other powers of 2 that appear are precisely those whose exponents are the prime numbers, and they appear in increasing order of magnitude ($2^1, 2^2, 2^3, 2^5, 2^7, 2^{11}, ...$).
Theorem 2: When PIGAME:

\[
\begin{array}{cccccccccccccccccc}
365 & 29 & 79 & 679 & 3159 & 83 & 473 & 638 & 434 & 89 & 17 & 79 \\
31 & 41 & 517 & 111 & 305 & 23 & 73 & 61 & 37 & 19 & 89 & 41 & 33 & 53 \\
183 & 115 & 89 & 83 & 79 & 73 & 71 & 67 & 61 & 59 & 57 & 47 & 43 \\
86 & 13 & 23 & 67 & 71 & 83 & 475 & 59 & 41 & 1 & 1 & 1 & 1 & 89 & 1 \\
41 & 38 & 37 & 31 & 29 & 19 & 17 & 13 & 291 & 7 & 11 & 1024 & 97 & 89 \\
\end{array}
\]

is started at \(2^n\), the next power of 2 to appear is \(2^{\pi(n)}\), where for

\[n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, \ldots\]

\[\pi(n) = 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, \ldots\]

For an arbitrary natural number \(n\), \(\pi(n)\) is the \(n\)th digit after the point in the decimal expansion of the number \(\pi\).

\[x_{t+1} = F(x_t) \quad t=0, 1, \ldots\]

[John H. Conway, "Unpredictable Iterations" 1972]
Fractran program

\[ \begin{align*}
17 & \quad 78 & \quad 19 & \quad 23 & \quad 29 & \quad 77 & \quad 95 & \quad 77 & \quad 1 & \quad 11 & \quad 13 & \quad 15 & \quad 1 & \quad 55 \\
\frac{1}{91} & \quad \frac{1}{85} & \quad \frac{1}{51} & \quad \frac{1}{38} & \quad \frac{1}{33} & \quad \frac{1}{29} & \quad \frac{1}{23} & \quad \frac{1}{19} & \quad \frac{1}{17} & \quad \frac{1}{13} & \quad \frac{1}{11} & \quad \frac{1}{2} & \quad \frac{1}{7} & \quad \frac{1}{1}
\end{align*} \]

Trajectory. \( 2, 3.5, 3.5^2.11, 5^2.29, 5^2.7.11, \ldots \)

Encoding. \( 2^{V_1}3^{V_2}5^{V_3}7^{V_4}p \quad p = 1 \) or \( 11, 13, 17, 19, 23, 29 \).

Equivalent program (4 registers \( V_i \geq 0 \) and 7 states).

<table>
<thead>
<tr>
<th>state</th>
<th>( V_1 )</th>
<th>( V_2 )</th>
<th>( V_3 )</th>
<th>( V_4 )</th>
<th>new state</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td></td>
<td></td>
<td>-1</td>
<td></td>
<td>17</td>
</tr>
<tr>
<td>17</td>
<td>+1</td>
<td>+1</td>
<td>-1</td>
<td></td>
<td>13</td>
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<tr>
<td>17</td>
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<td>-1</td>
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<td>19</td>
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<td>19</td>
<td>-1</td>
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<td>23</td>
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<td></td>
<td></td>
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<td>:</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>+1</td>
<td></td>
<td>11</td>
</tr>
</tbody>
</table>
Fractran is computationally universal

Associated to a computable function $f$ there is a Fractran program that, started from $2^n$, produces $2^{f(n)}$ as the next power of 2.

**Halting problem**

- **Instance**: Program $M$, input $x$ to $M$
- **Question**: $M$ halts on $x$?

- **Instance**: Fractran program $F$, initial $x_0$
- **Question**: The trajectory emanating from $x_0$ reaches 1 ($F^k(x_0) = 1$)?

- **Instance**: Fractran program $F$
- **Question**: The trajectory emanating from 1 returns to 1 ($F^k(1) = 1$)?

**Notation**: $F^k(x) = F(F(F(...F(x))))$
Outline

\[ x_{t+1} = F(x_t) \]  
**state** \( x_t \in \mathbb{R}^n \)  
\( t=0, 1, \ldots \)

Saturated systems  
\[ x_{t+1} = \sigma(A \cdot x_t) \]

Linear systems  
\[ x_{t+1} = A \cdot x_t \]

Switched systems  
\[ x_{t+1} = A_0 \cdot x_t \text{ or } A_1 \cdot x_t \]

Observability in graphs

Throughout the talk: open problems
Outline

\[ x_{t+1} = F(x_t) \quad \text{state } x_t \in \mathbb{R}^n \quad t=0, 1,\ldots \]

Saturated systems

\[ x_{t+1} = \sigma (Ax_t) \]
1. $V_1 \leftarrow V_1 - 1$
2. $V_2 \leftarrow V_2 + 1$
3. if $V_1 \neq 0$ goto 1
4. $V_2 \leftarrow V_2 - 1$
5. if $V_2 \neq 0$ goto 2

\[
\begin{pmatrix}
V_1 \\
V_2 \\
l_1 \\
l_2 \\
l_3 \\
l_4 \\
l_5
\end{pmatrix}
= \begin{pmatrix}
\sigma(V_1 - l_1) \\
\sigma(V_2 + l_2 - l_4) \\
\sigma(l_3 - \sigma(1 - V_1)) \\
\sigma(l_1 + \sigma(l_5 - \sigma(1 - V_2))) \\
\sigma(l_2) \\
\sigma(l_3 - \sigma(V_1)) \\
\sigma(l_5)
\end{pmatrix}
\]

\[
\sigma(x) = \max(0, x)
\]

\[
\begin{pmatrix}
V_1 \\
V_2 \\
l_1 \\
l_2 \\
l_3 \\
l_4 \\
l_5
\end{pmatrix}
\begin{pmatrix}
12 \\
4 \\
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
11 \\
4 \\
0 \\
1 \\
0 \\
0 \\
0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
11 \\
5 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}
\rightarrow \ldots
\begin{pmatrix}
0 \\
16 \\
0 \\
0 \\
0 \\
1 \\
0
\end{pmatrix}
\rightarrow \ldots
\]

\[
x_{k+1} = \sigma(A x_k)
\]
Saturated systems

Programs compute arbitrary computable functions

1. $V_1 \leftarrow V_1 - 1$
2. $V_2 \leftarrow V_2 + 1$
3. if $V_1 \neq 0$ goto 1
4. $V_2 \leftarrow V_2 - 1$
5. if $V_2 \neq 0$ goto 2

Saturated systems compute arbitrary computable functions

$X_{k+1} = \sigma (Ax_k)$

Halting problem

**Instance**: Program M, input x to M
**Question**: M halts on x?

Mortality problem

**Instance**: Program M
**Question**: M halts for every possible input and starting line?

Global convergence

**Instance**: Matrix A
**Question**: All trajectories reach the origin?

[Blondel, Bournez, Koiran, Papadimitriou, Tsitsiklis, 2001]
[Hooper, 1966]
Saturated and piecewise linear systems

Saturated system

\[ x_{t+1} = \sigma (A x_t) \]

Piecewise linear system

\[ \mathbb{R}^n = H_1 \cup H_2 \cup \ldots \cup H_m \]
\[ x_{t+1} = A_i x_t \text{ for } x_t \in H_i \]

**Instance**: Piecewise linear system, initial vector \( x_0 \)

**Question**: The trajectory emanating from \( x_0 \) goes to the origin

Undecidable, even for a partition in 800 pieces in dimension three \( \mathbb{R}^3 \)
A piecewise-linear function $F$ on the unit interval and $x$ a point in this interval. Does the trajectory emanating from $x$ reach a fixed point?

[P. Koiran, my favourite problems, 2007]
Outline

\[ x_{t+1} = F(x_t) \]

state \( x_t \in \mathbb{R}^n \)

t = 0, 1, ...

Saturated systems

\[ x_{t+1} = \sigma(Ax_t) \]

Linear systems

\[ x_{t+1} = Ax_t \]
Linear system

\[ x_{t+1} = A x_t \quad x_t \in \mathbb{R}^n \]

**Global convergence to the origin.** The iterates \( A^k x_0 \) converge to the origin? Decidable

**Point-to-point.** Given \( x_0 \) and \( x_* \), is there a \( k \) for which \( x_* = A^k x_0 \)? Decidable

**Pisot or Skolem’s problem**

**Point-to-subspace.** Given \( A \), \( x_0 \) and \( c \), is there a \( k \) for which \( c^T A^k x_0 = 0 \)? Decidable or not?

**Equivalent problem:** Does a given linear recurrence have a zero?

\[
\begin{align*}
x_{n+1} &= 3 x_n - 7 x_{n-1} + 6 x_{n-2} - 2 x_{n-3} \\
x_0 &= 2, \ x_1 = -1, \ x_2 = 3, \ x_3 = 1
\end{align*}
\]

[Blondel, Portier, 2002]
Outline

\[ x_{t+1} = F(x_t) \]

state \( x_t \in \mathbb{R}^n \) \( t=0, 1,... \)

Saturated systems

\[ x_{t+1} = \sigma(A \ x_t) \]

Linear systems

\[ x_{t+1} = A \ x_t \]

Switched systems

\[ x_{t+1} = A_0 \ x_t \text{ or } A_1 \ x_t \]
Switched systems

\[ x_{t+1} = \begin{cases} A_0 \ x_t \\ A_1 \ x_t \end{cases} \]

\[ x_T = A_0 \ A_0 \ A_1 \ A_0 \ldots \ A_1 \ x_0 \]

**Point-to-point.** Given \( x_0 \) and \( x_\ast \), is there a product of the type \( A_0 \ A_0 \ A_1 \ A_0 \ldots \ A_1 \) for which \( x_\ast = A_0 \ A_0 \ A_1 \ A_0 \ldots \ A_1 \ x_0 \)?
**Post correspondence problem**

**Instance:** Pairs of words

\[ U_1 = 1 \quad V_1 = 12 \]

\[ U_2 = 1212 \quad V_2 = 12 \]

\[ U_3 = 2 \quad V_3 = 1 \]

**Question:** is a correspondence possible?

\[ U_1 \ U_3 \ U_1 \ U_2 \quad V_1 \ V_3 \ V_1 \ V_2 \]

\[ 1211212 \quad 1211212 \]

Decidable for 2 pairs, undecidable for 7

[Matiyasevich, Senizergues, 1996]

\[ \begin{bmatrix} 10 & 0 & 0 \\ 0 & 100 & 0 \\ 1 & 12 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 10000 & 0 & 0 \\ 0 & 100 & 0 \\ 1212 & 12 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 2 & 1 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 12121 & 1212 & 1 \end{bmatrix} \]

[Paterson, 1970]
Switched systems

\[ x_{t+1} = \begin{cases} A_0 x_t \\ A_1 x_t \end{cases} \]

\[ x_T = A_0 A_0 A_1 A_0 \ldots A_1 x_0 \]

Point-to-point. Given \( x_0 \) and \( x_* \), is there a product of the type \( A_0 A_0 A_1 A_0 \ldots A_1 \) for which \( x_* = A_0 A_0 A_1 A_0 \ldots A_1 x_0 \)?

Global convergence to the origin. Do all products of the type \( A_0 A_0 A_1 A_0 \ldots A_1 \) converge to zero?
Global convergence

**Input.** Matrices $A_0$, $A_1$

**Question:** Do all products of the type $A_0 A_0 A_1 A_0 \ldots A_1$ converge to zero?

The **spectral radius** of a matrix $A$ controls the growth or decay of powers of $A$

$$\rho(A) = \lim_{k \to \infty} \|A^k\|^{1/k}$$

The powers of $A$ converge to zero iff $\rho(A) < 1$

The **joint spectral radius** of $A_0$ and $A_1$ is given by

$$\rho(A_0, A_1) = \lim_{k \to \infty} \max_{i_1, \ldots, i_k} \|A_{i_1} \cdots A_{i_k}\|^{1/k}$$

All products of $A_0$ and $A_1$ converge to zero iff $\rho(A_0, A_1) < 1$
Joint spectral radius: everywhere

\[ \rho(A_0, A_1) = \lim_{k \to \infty} \max_{i_1, \ldots, i_k} \|A_{i_1} \cdots A_{i_k}\|^{1/k} \]

[Rota, Strang, 1960]

Gil Strang, 2001: Every few years, Gian-Carlo Rota would ask me whether anyone ever read our paper. After I had tenure, I could tell him the truth: "not often". In recent years I could change my answer!

Wavelets (continuity of wavelets) 1992
Control theory (hybrid systems), 1980+
Curve design (subdivision schemes) 1990+
Autonomous agents (consensus rate) 1990+
Number theory (asymptotics of the partition function), 2000
Coding theory (constrained codes), 2001
Sensor networks (trackability), 2005
Etc…
Finiteness conjecture

[Lagarias and Wang 1995]: The asymptotic rate of growth of products of two matrices can always be obtained for a periodic product

\[
\begin{bmatrix}
3 & 0 \\
1 & 3
\end{bmatrix}
\begin{bmatrix}
3 & -3 \\
0 & -1
\end{bmatrix}
\]

\[3.298 \leq \rho \leq 3.351\]

If the finiteness conjecture is true, then \(\rho(A_0, A_1) < 1\) is decidable.
**Theorem.** In a heap of two pieces, a minimal growth rate can always be obtained with a Sturmian sequence.

[Gaubert, Mairesse, 1999]
Sturmian sequence

The infinite sequence $\bullet \bullet \bullet \bullet \bullet \bullet$ is a sturmian sequence. If the slope is rational, the sequence is periodic, otherwise it is not.
Theorem. In a heap of two pieces, a minimal growth rate can always be obtained with a Sturmian sequence.

[Gaubert, Mairesse, 1999]
Finiteness conjecture

[Lagarias and Wang 1995]: The asymptotic rate of growth of products of two matrices can always be obtained for a periodic product

**Theorem.** Let $a$ be a scalar. The optimal rate of growth of the matrices

$$
\begin{bmatrix}
1 & 0 \\
1 & 1 \\
\end{bmatrix} \quad a \quad \begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
$$

can always be obtained by a Sturmian product sequence. There are values of $a$ for which this sequence is not periodic.

[Blondel, Theys, Vladimirov, 2003]
[Blousch, Mairesse, 2002]
Periodic optimality in graphs

The asymptotic rate of growth can not always be obtained with a periodic product.

But perhaps this is always possible for matrices with binary entries?

```
\begin{align*}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\end{align*}
```

Equivalent problem: In a bicolored graph, count the total number of paths that are allowed by a given color sequence. Can the largest possible rate of growth of this total number of paths always be obtained by a periodic sequence?
Outline

\[ x_{t+1} = F(x_t) \]

**state** \( x_t \in \mathbb{R}^n \) \( t=0, 1,... \)

- Saturated systems
  \[ x_{t+1} = \sigma(A x_t) \]

- Linear systems
  \[ x_{t+1} = A x_t \]

- Switched systems
  \[ x_{t+1} = A_0 x_t \text{ or } A_1 x_t \]

- Observability in graphs
Control theory

\[ x_{t+1} = f(x_t, u_t) \]
\[ y_t = g(x_t) \]

**Observability**: observe \( y \), construct \( x \)

**Controllability**: choose \( u \) to drive the state \( x \)
**Observable.** Where am I in the graph?

**Controllable (Synchronizing).** Can I choose a color sequence that drives me to a particular node?
Colors on nodes, colors on edges
A graph is **observable** if there is some $K$ for which the position in the graph can always be determined after an observation of length at most $K$. 
A graph is **observable** if there is some $K$ for which the position in the graph can always be determined after an observation of length at most $K$.

**Theorem.** These necessary conditions are also **sufficient** for a graph to be observable. Moreover, the conditions can be **checked in polynomial time**.

If the graph is observable then the position in the graph can be determined after an **observation of length at most $n^2$** ($n =$ number of nodes).

[Jungers, Blondel, 2006]

Observable DES [Ozveren, Willsky, 1990], local automata [Beal, 1993]
Proof (sketch)
Making a graph observable

Theorem. The problem of determining the minimal number of node colors needed to make a graph cruisable is a problem that is NP-hard.

[Jungers, Blondel, 2006]
Synchronizing graphs

Graphs that have one outgoing edge of every color from every node

Synchronizing sequence (or reset sequence)

A graph is **synchronizing** if it has a synchronizing sequence, i.e., there is a node $x$ and a color sequence that leads all paths with that color to $x$.

[Cerny, 1960's]
Controlling a robot

Do ⬤ ⬤ ⬤ ⬤ and you are at 5…
Cerny’s conjecture (1964). If a graph is synchronizing, then it admits a synchronizing sequence of length at most \((n-1)^2\).

- [1964 Cerny] \(2^n - n - 1\)
- [1966 Starke] \(n^3/2 - 3/2 n^2 + n + 1\)
- [1970 Kohavi] \(n(n-1)^2/2\)
- [1978 Pin] \(7/27 n^3 - 17/18 n^2 + 17/6 n - 3\)
- [1982 Frankl] \((n^3 - n)/6\)
- [1990 Eppstein] Monotonic automata
- [1998 Dubuc] Circular automata
- [2001 Kari] Eulerian graphs
Graphs and matrices

Adjacency matrix $A$

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
$$

$(A^k)_{ij} =$ number of paths of length $k$ between nodes $i$ and $j$
Colored graphs

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}$$

Adjacent matrix $$A$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Adjacent matrix $$A$$

$$(A A A A)_{ij} = \text{number of paths of color 0000 between } i \text{ and } j$$
Cerny’s conjecture. Let $A$ and $A$ be matrices with exactly one 1 in every row. If there is a product of $A$ and $A$ for which all 1’s are in the same column, then there is such a product of length at most $(n-1)^2$.
Assign colors to the edges so that the resulting graph is synchronizing

**Road coloring conjecture**: Can always be done provided the graph is aperiodic. 

[Adler et al., 1977]
Conclusions

\[ x_{t+1} = F(x_t) \]

**state** \( x_t \in \mathbb{R}^n \) \( t=0, 1, \ldots \)

- Saturated systems
- Linear systems
- Switched systems
- Observability in graphs

\( x_{t+1} = \sigma(A \ x_t) \)
\( x_{t+1} = A \ x_t \)
\( x_{t+1} = A_0 \ x_t \ or \ A_1 \ x_t \)

*Promenade* parmi quelques problèmes en systèmes dynamiques discrets
Fractran, correspondance de Post, problème de Pisot, sturmiens, rayon spectral conjoint, conjecture Cerny, ...

References: Google(Vincent Blondel)