Application of Game Theory to Wireless Networking

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Outline

Objectives of this presentation

- Present a general game theoretic framework for distributed control under limited information exchange.
- Illustrate the game theoretic approach via a specific application: uplink power control in wideband wireless networks.
- Investigate existence and uniqueness of Nash equilibrium.
- Convergence and stability analysis of continuous-time distributed algorithms.
- Study of relevant distributed iterative (update) algorithms and their convergence conditions to the equilibrium.

Network Games

- Game theory (GT) involves multi-person decision making.
- Autonomous parts of the networked systems (such as mobiles, devices generating Internet traffic etc.) are modeled as players.
- Players interact and compete with each other on the same system for limited and shared resources: e.g. quality of service, bandwidth...
- Players are associated with cost functions, which they minimize by choosing a strategy from a well defined strategy space.
- Nash equilibrium (NE) provides an appropriate solution concept, which is (approximately) optimal w.r.t. a global objective function.

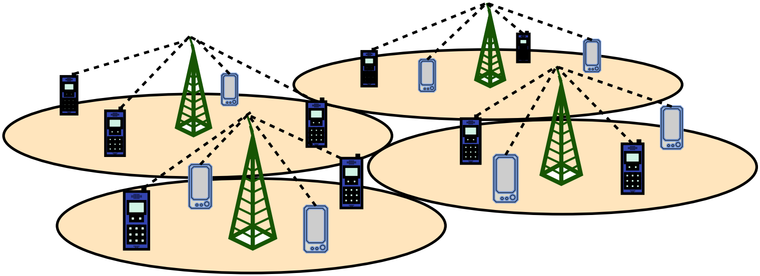
Why Game Theory

- The microprocessor revolution enabled production of systems with significant processing capacities $\rightarrow$ independent decision makers.
- These system are connected to each with a variety wired/wireless communication technologies resulting in networked systems $\rightarrow$ interaction between decision makers.
- The systems share various resources (but often have only local information) $\rightarrow$ competition for available resources (resource allocation).

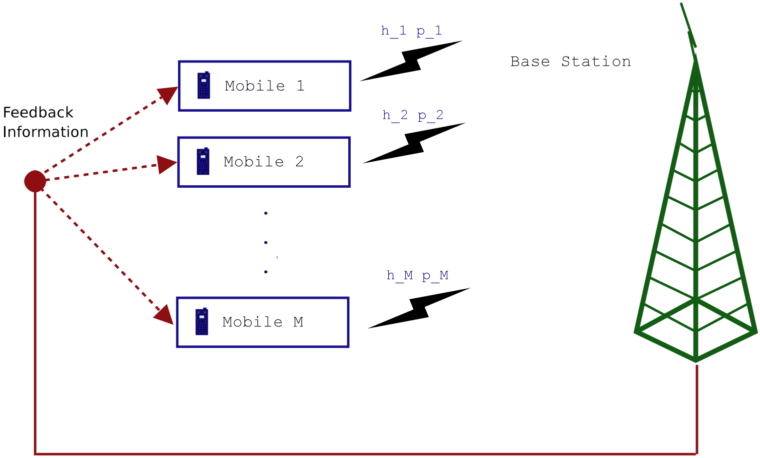
Uplink Power Control in Wireless Networks

- Primary objective of (uplink) power control is to regulate the transmission power level of each mobile in order to obtain and maintain a satisfactory quality of service or Signal-to-interference ratio (SIR) level.
- In wideband systems such as CDMA, signals of the users interfere and affect each other's service (SIR) level.
- In data networks, unlike in voice communication, SIR requirements vary from one user to another.
- Emerging technologies such as cognitive radio empowers mobile users with independent decision making capabilities.

A Multicell Wireless Network



Distributed Power Control



Game Theoretic Formulation

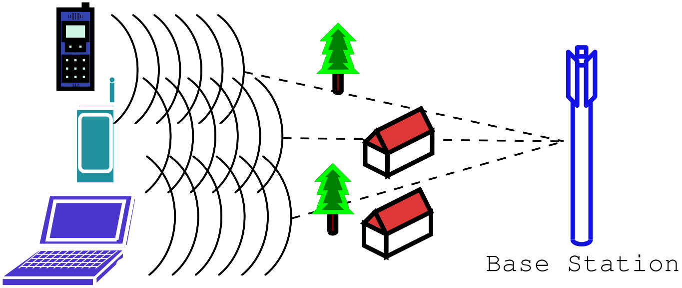
- Game theory provides a natural framework for power control in wireless systems, where mobiles (players) compete for service quality: e.g. cognitive radio.
- A mobile has no information on other player's power level or preferences. Therefore, use of noncooperative game theory is appropriate.
- Existence of a unique Nash equilibrium (NE) point is established in this multicell power control game.
- Convergence of continuous and discrete-time synchronous and asynchronous update schemes as well as of a stochastic update scheme is investigated.
- The power control game and the update algorithms are demonstrated through numerical simulations.

Network Model

- The system consists of $\mathcal{L} := \{1, \ldots, \bar{L}\}$ cells, with M_l users in cell l.
- Define $0 < h_{ij} < 1$ as the channel gain. Let secondary interference effects from neighboring cells be modeled as background noise, of variance σ^2.
- The i^{th} mobile transmits with an uplink power level of $p_i \leq p_{i,max}$, which is received at the BS j as $x_{ij} := h_{ij} p_i$. Then, SIR obtained by mobile i is given by

$$\gamma_{ij} := \frac{L h_{ij} p_i}{\sum_{k \neq i} h_{kj} p_k + \sigma^2}$$

Network Model



Cost Function

- Each mobile is associated with a cost function:

$$J_i(x_i, \mathbf{x}_{-i}, h_i) = P_i(x_i) - U_i(\gamma_i(\mathbf{x}))$$

- The benefit (utility) function, $U_i(\gamma_i)$ quantifies the user demand for quality of service or SIR level.
- The "pricing" function, $P_i(p_i)$ is imposed to limit the interference, and hence, improve the system performance. It can also be interpreted as a cost on the battery usage.
- Terminology clarification:

$$\begin{aligned} \max \quad & \textit{Payoff} = \textit{Benefit} - \text{``}\textit{Cost}\text{''} \\ \min \quad & \textit{Cost} = -\textit{Utility} + \textit{Price} \end{aligned}$$

Nash Equilibrium (NE)

Definition

The Nash equilibrium is defined as a set of strategies (and corresponding set of costs), with the property that no player can benefit by modifying its own strategy while the other players keep theirs fixed.

If $\mathbf{x}$ is the strategy vector of players and X is the strategy space such that $\mathbf{x} \in X \ \ \forall \mathbf{x}$, then $\mathbf{x}^*$ is in NE when x_i^* of any i^{th} player satisfies

$$\min_{x_i} J_i(x_i, \mathbf{x}^*_{-i}),$$

where J_i is the cost function of the i^{th} player and $\mathbf{x}^*_{-i}$ is the equilibrium strategies of all other players.

NE of a Generic Noncooperative Game

Assumptions:
A1 The strategy space X of a noncooperative game, Θ is convex, compact, and has a nonempty interior, $X^o \neq \varnothing$.

A2 The cost function of the i^{th} player, $J_i(\mathbf{x})$, is twice continuously differentiable in all its arguments and strictly convex in x_i, i.e. $\partial^2 J_i(\mathbf{x})/\partial x_i^2 \geq 0$.

Let $\overline{\nabla}$ be the pseudo-gradient operator:
$\overline{\nabla} J := \left[\nabla_{x_1} J_1(\mathbf{x})^T \cdots \nabla_{x_M} J_M(\mathbf{x})^T\right]^T$.

Let in addition $G(\mathbf{x})$ be the Jacobian of $\overline{\nabla} J$ with respect to $\mathbf{x}$:

$$G(\mathbf{x}) := \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1M} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & b_M \end{pmatrix}_{M \times M}$$

where $b_i := \frac{\partial^2 J_i(\mathbf{x})}{\partial x_i^2}$ and $a_{i,j} := \frac{\partial^2 J_i(\mathbf{x})}{\partial x_i \partial x_j}$.

We also define the symmetric matrix $\mathcal{G}(\mathbf{x}) := G(\mathbf{x}) + G(\mathbf{x})^T$.

Proposition

*The strategy vector $\mathbf{x}^* \in X^o$ is an inner NE solution of the game Θ, if assumptions **A1** and **A2** hold, and $\overline{\nabla} J(\mathbf{x}^*) = 0$. In addition, if $\mathcal{G}(\mathbf{x})$ is positive definite for all $\mathbf{x}$ then there can be at most one inner NE solution in the game Θ. Furthermore, under **A1**, Θ admits a NE.*

Notice that, this condition is quite similar to the strict convexity condition where Hessian of a multivariable function $f(x_1, \ldots, x_n)$ is required to be positive definite:

$$H(f) := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_M} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_M \partial x_1} & \frac{\partial^2 f}{\partial x_M \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_M^2} \end{pmatrix}_{M \times M}$$

A3 Let $X := \{\mathbf{x} \in \mathbb{R}^M : h_j(\mathbf{x}) \leq 0, \forall j\}$, where $h_j : \mathbb{R}^M \to \mathbb{R}, \forall j$, $h_j(\mathbf{x})$ is convex in its arguments for all j, and the set X is bounded and has a non-empty interior. In addition, the derivative of at least one of the constraints with respect to x_i, $\{dh_j(\mathbf{x})/dx_i, \forall j\}$, is nonzero for $i = 1, 2, \ldots M, \;\; \forall \mathbf{x} \in X$.

The Lagrangian function for player i in this game is given by $L_i(\mathbf{x}, \mu) = J_i(\mathbf{x}) + \sum_{j=1}^{r} \mu_{i,j} h_j(\mathbf{x})$.

Theorem

There exists a unique NE point in the M-player noncooperative game Θ *if* ***A1***, ***A2***, *and* ***A3*** *hold.*

Theorem

Under appropriate convexity conditions on the cost functions J *the multicell power control game defined admits a unique inner Nash equilibrium solution.*

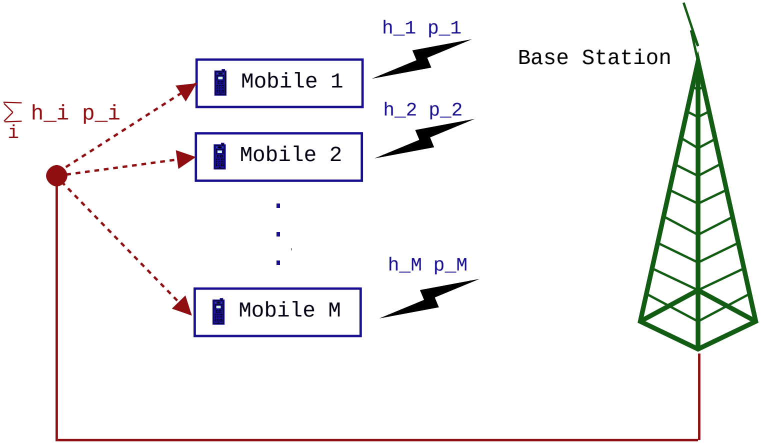
System Dynamics and Stability

- Each mobile uses a gradient algorithm to solve its own optimization problem. The update scheme of the i^{th} mobile is:

$$\dot{p}_i = \frac{dp_i}{dt} = -\lambda_i \frac{\partial J_i}{\partial p_i}$$

- In terms of the received power level, x_i, at the BS:

$$\dot{x}_i = \frac{dU_i}{d\gamma_i} \frac{L \lambda_i h_i^2}{\sum_{j \neq i} x_j + \sigma^2} - \lambda_i h_i \frac{dP_i}{dp_i} := \phi_i(\mathbf{x}).$$


∑ h_i p_i
i
Mobile 1
Mobile 2
Mobile M
h_1 p_1
h_2 p_2
h_M p_M
Base Station

Stability in a Cell

Define the quadratic and radially unbounded Lyapunov function

$$V_l := \sum_{i \in \mathcal{M}_l} \phi_i^2(\mathbf{x})$$

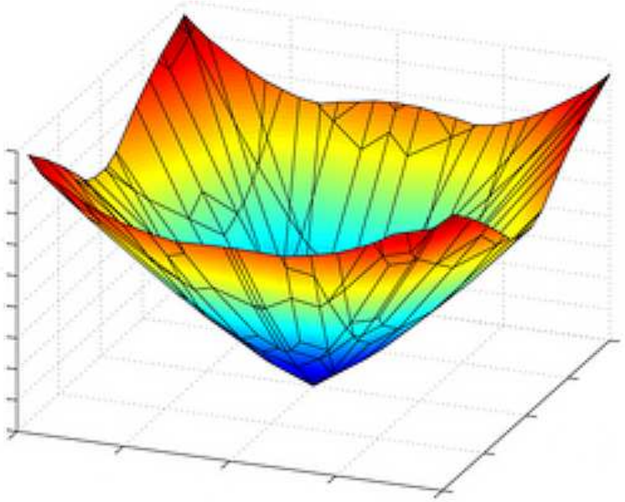
A sufficient condition for $\dot{V}_l < 0$, uniformly in the x_i's, is

$$L > M_l - 1$$

in the symmetric case where $U_i = U_j$ and $x_i = x_j \;\; \forall i, j \in \mathcal{M}_l$, and for a large class of logarithmic utility functions of the form $U_i = u_i log(k\gamma_i + 1)$.

Then, the distributed power update scheme is globally asymptotically stable!

Lyapunov Function (representation)



Outage Probability

- The outage probability of user i, denoted O_{il}, is defined as the proportion of time that some SIR threshold, $\bar{\gamma}_{il}$, is not met for sufficient reception at the l^{th} BS receiver
- By a careful choice of $\bar{\gamma}_{il}$, a quality of service level can be established for each user. Assume $\bar{\gamma}_i := \bar{\gamma}_{il} = \bar{\gamma}_{ik}\ \forall l, k \in \mathcal{L}$ as a simplification.
- The outage probability, $O_i = Pr(\gamma_i \le \bar{\gamma}_i)$, of the i^{th} mobile is

$$O_i(\mathbf{x}, \bar{\gamma}_i) = 1 - \exp\left(\frac{-\sigma^2 \bar{\gamma}_i}{x_i}\right) \Pi_{j \neq i} \frac{1}{1 + \frac{\bar{\gamma}_i x_{jl}}{x_i}}.$$

Outage-Based Cost Function

- Each mobile is associated with a cost function:

$$J_i(\mathbf{x}) = P_i(x_i) - U_i(Pr_i(\gamma_i(\mathbf{x}) \geq \bar{\gamma}_i),$$

  where $Pr_i(\gamma_i(\mathbf{x}) \geq \bar{\gamma}_i) = 1 - O_i(\mathbf{x}, \bar{\gamma}_i)$. Hence, $U_i = u_i \log(1 - O_i(\mathbf{x}, \bar{\gamma}_i))$.
- The utility function, $U_i(Pr_i(\gamma_i(\mathbf{x}) \geq \bar{\gamma}_i)$ quantifies the user demand for a certain level of service or **outage probability**.

Theorem

Under certain convexity assumptions, the multicell power control game defined admits a unique inner Nash equilibrium solution.

Synchronous Update

Consider a discrete-time update scheme in a system with M mobiles where each mobile uses a discretized gradient algorithm to solve its optimization problem:

$$p_i(n+1) = p_i(n) - \lambda_i \frac{\partial J_i}{\partial p_i} \ \ \forall i \in \mathcal{M} \ ,$$

where $n = 1, 2, \ldots,$ denotes the update instances and λ_i is the user-specific step size constant.
This can also be defined as

$$x_i(n+1) = T_i(\mathbf{x}(n)) := x_i(n) - \lambda \frac{\partial J_i}{\partial x_i} \ \ \forall i \in \mathcal{M} \ .$$

Synchronous Update

Theorem

Let $x_{max} = \alpha x_{min}$ *for some* $\alpha > 0$ *and* $X := \{\mathbf{x} \in \mathbb{R}^M x : x_{min} \leq x_{il} \leq x_{max} \ \forall i, l\}$. ***The synchronous power update algorithm***

$$p_i(n+1) = p_i(n) - \lambda_i \frac{\partial J_i}{\partial p_i} \ \ \forall i \in \mathcal{M}$$

converges to the unique NE point of the power control game, $p^* := [x_1^*/h_1, \ldots, x_M^*/h_M]$, *on the set* X ***if***

$$\lambda K_{synch} < 1,$$

and

$$\alpha < 1 + \sqrt{1 + \bar{\gamma}_{min}},$$

where K_{synch} *is a function of the system parameters and constant.*

Asynchronous Power Update

- A natural generalization of the synchronous update is the asynchronous update scheme.
- It is more realistic since it is difficult for the mobiles to synchronize their exact power update instances in a practical implementation.
- In this particular case, the convergence analysis above also applies to the asynchronous update algorithm.

Asynchronous Power Update

Define a sequence of nonempty, convex, and compact sets

$$X(k) := [x_1^* - \delta(k), x_1^* - \delta(k)] \times [x_2^* - \delta(k), x_2^* - \delta(k)] \times \ldots [x_M^* - \delta(k), x_M^* - \delta(k)],$$

where $\delta(k) := \|\mathbf{x}(k) - \mathbf{x}^*\|$. By the previous Theorem, $\delta(k+1) < \delta(k)$, we have

$$\ldots \subset X(k+1) \subset X(k) \subset \ldots X.$$

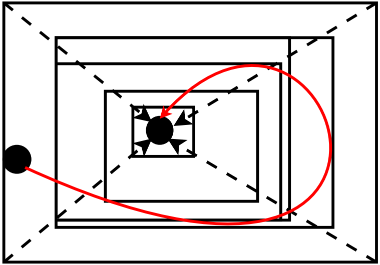
Asynchronous Power Update

Definition 1 [Synchronous Convergence Condition] For a sequence of nonempty sets $\{X(k)\}$ with $\ldots \subset X(k+1) \subset X(k) \subset \ldots X$, we have $T(\mathbf{x}) \in X(k+1), \ \forall k, \$ and $\mathbf{x} \in X(k)$. Furthermore, if $\{y^k\}$ is a sequence such that $y^k \in X(k)$ for every k, then every limit point of $\{y^k\}$ is a fixed point of T.

Definition 2 [Box Condition] For every k, there exist sets $X_i(k) \subset X_i$ such that

$$X(k) := X_1(k) \times X_2(k) \times \cdots \times X_M(k).$$

Asynchronous Power Update



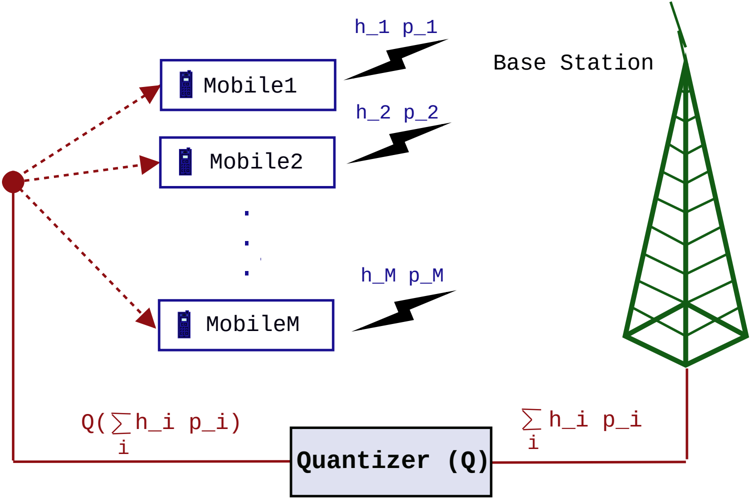
Both of these conditions are satisfied in this case by definition of $X(k)$ and synchronous convergence theorem.

Therefore, it immediately follows from asynchronous convergence theorem [Bertsekas] that **the asynchronous power update algorithm converges to the unique NE point of the power control game.**

Stochastic Power Update

- In a real life implementation, communication constraints, approximations, estimation and quantization errors are not negligible.
- Hence, a mobile does not have access to the exact values of the system parameters such as its own channel gain or the feedback terms provided by the BS.
- These uncertainties can be captured by defining a stochastic update algorithm for analysis purposes.

Communication Constraints



Stochastic Power Update

For each $i \in \mathcal{M}$, let $\xi_i(n)$ $n = 1, 2, \ldots$ be a sequence of independent identically distributed (iid) random variables defined on the common support set $[1-\varepsilon, 1+\varepsilon]$, where $0 < \varepsilon < 1$.

We further assume the sequence ξ_i is independent of the past of ξ_j, $j \neq i$.

Using these random sequences, we model the aggregate uncertainty in the term $\partial J_i / \partial p_i$ due to quantization, estimation, and multiplicatively approximation errors. The stochastic counterpart of the synchronous update algorithm is given by

$$\boxed{p_i(n+1) = p_i(n) - \lambda_i \xi_i(n) \frac{\partial J_i}{\partial p_i} \quad \forall i \in \mathcal{M}.}$$

Stochastic Power Update

This can also be described in terms of received power levels at the base station as

$$x_i(n+1) = x_i(n) - \lambda \xi_i(n) \frac{\partial J_i}{\partial x_i}$$

$$=: T_i(\mathbf{x}(n); \xi_i(n)) \;\; \forall i \in \mathcal{M}.$$

Let $\mathbf{x}_i(n)$ $(\xi_i(n))$ be random (random iid) sequences for all i, where ξ_i is associated with the probability density function $f_{\xi_i}(\xi_i)$ defined on the support set $[1-\varepsilon, 1+\varepsilon]$, $0 < \varepsilon < 1$, and the random vector $\mathbf{x}$ takes its values on the set $X := \{\mathbf{x} \in \mathbb{R}^{Mx} : x_{min} \leq x_{il} \leq x_{max} \; \forall i, l\}$. Furthermore, let $\alpha > 0$ be defined as $\alpha := x_{max}/x_{min}$.

Stochastic Power Update

Theorem

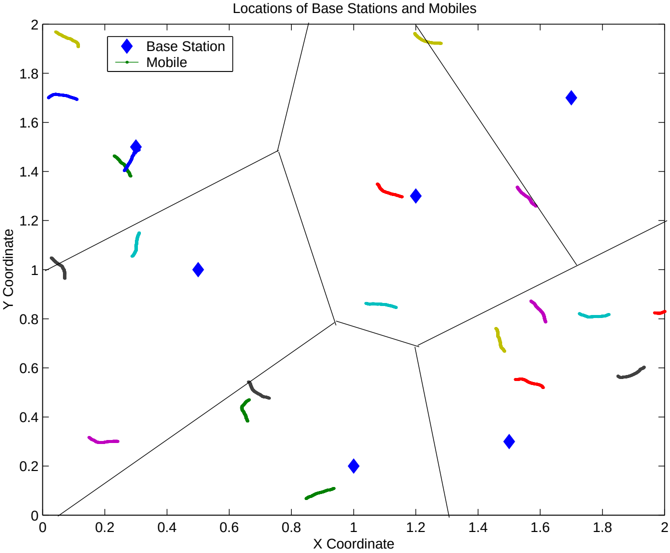
The stochastic power update algorithm converges almost surely to the unique NE point of the power control game, p^*, ***if***

$$\alpha < \frac{1}{2}\sqrt{\bar{\gamma}_{min}} + \frac{1}{4}$$

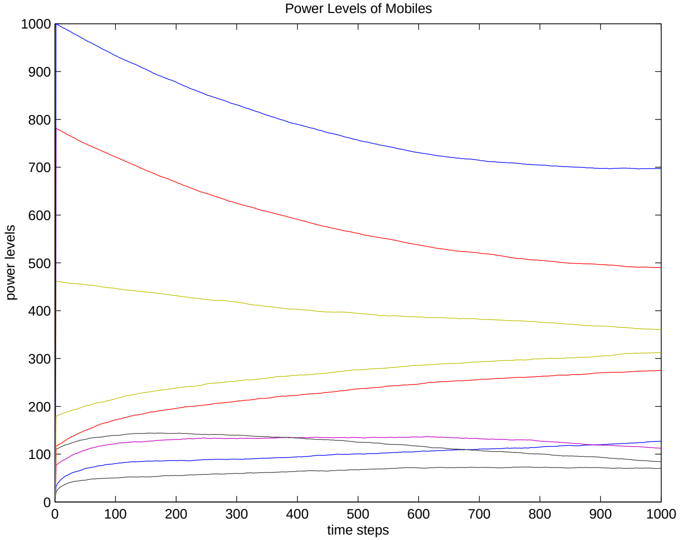
and

$$\lambda(1+\varepsilon)K_{sto} < 1$$

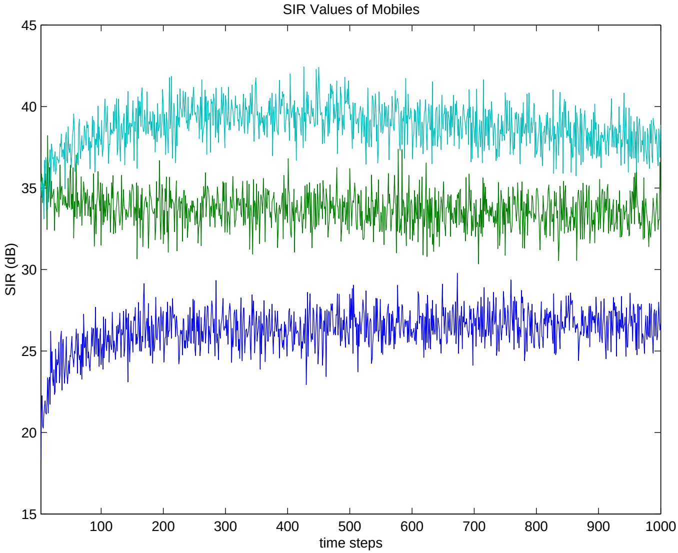
hold. Here K_{sto} *is a function of the system parameters and constant.*



Locations of base stations and the paths of mobiles.



Power levels of selected mobiles with respect to time.



SIR values of selected mobiles (in dB) versus time.

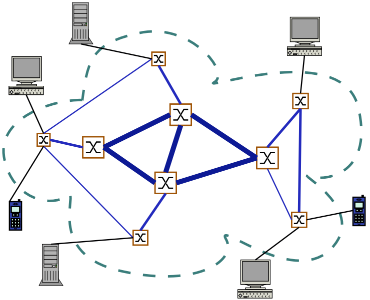
Conclusion

- We have considered a noncooperative power control game with a utility defined as the function of the the SIR level or outage probability.
- We have proved that this game admits a unique Nash equilibrium for uniformly strictly convex pricing functions and/or under some technical assumptions on the SIR threshold levels.
- We have established the global convergence of continuous-time as well as discrete-time synchronous, asynchronous, and stochastic iterative power update algorithms to the unique NE of the game under some conditions.
- Finally, through simulation studies we have demonstrated the convergence and robustness properties of power update schemes developed.

Conclusion

- We have presented a game theoretic framework for distributed control where individual parties (clients, mobile devices, etc.) compete for resources and have limited information.
- We have established conditions for existence and uniqueness of Nash equilibrium in the resulting game.
- We have studied convergence and stability properties of continuous-time as well as discrete-time distributed algorithms.
- Using two example power control games in the context of wireless networks, we have illustrated the framework presented.

Congestion Control Problem



- The users communicate with each other on the network by sharing the available bandwidth.
- The bandwidth becomes congested as a resource when the total demand exceeds the capacity.

The problem is complicated by communication constraints such as communication delays, distributed nature of the system, and users requesting as much bandwidth as possible.

Congestion Control Game

The Network Fluid approximation model. N nodes and L links with capacities C_l. M users, each associated with a (unique) connection. Routes are fixed and described by the routing matrix $\mathbf{A}$.

Flow Rates: User i has a nonnegative flow rate x_i. Flows satisfy the capacity constraint $\mathbf{Ax} \leq \mathbf{C}$.

Cost Function: Each user is associated with a cost function

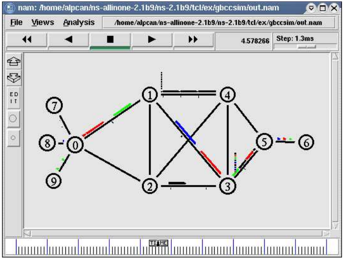
$$J_i(\mathbf{x};\mathbf{C},\mathbf{A}) = P_i(\mathbf{x};\mathbf{C},\mathbf{A}) - U_i(x_i), i \in \mathbf{M}.$$

◇ The function P acts as a "feedback" term indicating the state of the network.
◇ The function U models the user's demand for bandwidth.

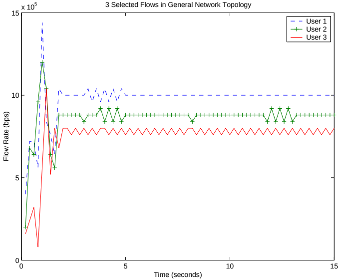
Congestion Control: Overview of Results

- Developed a general framework for study of network congestion control based on game theory.
- Developed distributed, end-to-end congestion control algorithms and analyzed their stability and delay robustness properties both theoretically and numerically.
- Utilized randomized algorithms to investigate stability of discrete-time nonlinear algorithms in cases where analytical models are intractable.
- Verified theoretical results obtained through both numerical and realistic packet level simulations.

Simulation Results



A *Nam* screenshot of the general (arbitrary) topology network.



Three flows from nodes 7, 8, and 9 to node 6 are shown where these users are symmetric.

Merci!

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or

http://deutsche-telekom-laboratories.de/~alpcan/